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Withdrawal from a fluid of finite depth through a line sink, including surface-tension effects

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Abstract. The steady withdrawal of an inviscid fluid of finite depth into a line sink is considered for the case in which surface tension is acting on the free surface. The problem is solved numerically by use of a boundary-integral-equation method. It is shown that the flow depends on the Froude number, $F_B = m(gH_B^3)^{-1/2}$, and the nondimensional sink depth $\lambda = H_S/H_B$, where m is the sink strength, g the acceleration of gravity, H_B is the total depth upstream, H_S is the height of the sink, and on the surface tension, T. Solutions are obtained in which the free surface has a stagnation point above the sink, and it is found that these exist for almost all Froude numbers less than unity. A train of steady waves is found on the free surface for very small values of the surface tension, while for larger values of surface tension the waves disappear, leaving a waveless free surface. It the sink is a long way off the bottom, the solutions break down at a Froude number which appears to be bounded by a region containing solutions with a cusp in the surface. For certain values of the parameters, two solutions can be obtained.

Key words: free-surface flow, selective withdrawal, water waves, line sink, surface tension.

1. Preliminary discussion

There are a number of problems involving withdrawal of water from lakes or reservoirs, either from single layers or from one of several layers of different density which impinge on such questions as water quality and reservoir management [1]. It has long been known that when water is withdrawn from a basin containing several layers of different density, the water flows from the layer adjacent to the outlet until some critical flow rate is exceeded, after which water flows from both (or several) layers (see *e.g.* [2–7]). Experiments and semi-analytical and numerical solutions of this problem have given different values for this critical withdrawal flow rate, often with quite a large variation in values for the same geometric configuration. This is true in the case of both two-dimensional withdrawal (slot or line sink), and in the case of three-dimensional withdrawal (pipe or line sink). The reason for these discrepancies is still unclear, despite a great deal of work.

The flow under consideration can be characterized by the Froude number

$$F_{SP} = \frac{m}{\sqrt{gH^3}}.$$

Here m is the strength of the line sink (*i.e.* the flux in the far field), g is gravity, H is the distance between the bottom of the fluid and the stagnation level of the free surface in the

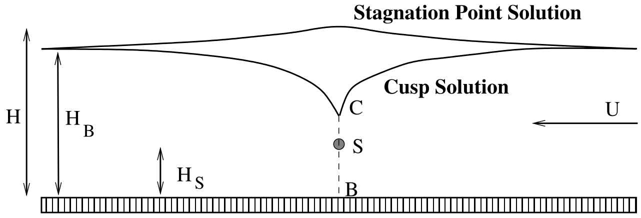

Figure 1. The two main free-surface shapes for flow into a line sink from a layer of finite depth.

absence of surface tension. The average upstream depth is given by $H_B = H(1-\gamma)$, so that the usual depth-based free-stream Froude number is

$$F_B = \frac{F_{SP}}{(1-\gamma)^{3/2}}.$$

The Froude number F_B determines whether or not there can be waves on the free surface. The linear theory of water waves shows that, if $F_B < 1$, then waves can occur on the free surface of a running stream, while if $F_B \geqslant 1$, waves are not possible. Flows with $F_B < 1$ and $F_B > 1$ are called subcritical and supercritical, respectively, for this reason. This distinction is based on linear theory. The terms subcritical and supercritical are also commonly used in nonlinear theory, although nonlinear waves are possible, not only for $F_B < 1$, but also for $1 < F_B < 1{\cdot}3$ (the value 1·3 corresponds to the highest solitary wave).

Solutions of the steady flow with a cusp on the interface or free surface [2, 8–12] (see 'Cusp solution' in Figure 1) have long been thought to correspond to the critical drawdown value; if the flow rate was increased, the water above the interface would begin to flow out. Tuck and Vanden-Broeck [11] obtained numerically such a 'cusp solution' for a line sink in water of infinite depth (*i.e.* $H \to \infty$). They found a unique solution, at $F_S = 12{\cdot}622$, where $F_S^2 = m^2/gD_S^3$ is a Froude number based on the sink depth, D_S. Hocking [13] recently provided strong evidence that this solution is the critical value for this case of 'infinite' depth. As the flow rate was decreased, the solutions obtained approached the single-layer solutions of Tuck and Vanden-Broeck [11]. In addition to these papers Hocking [8] computed solutions, similar to those of Tuck and Vanden-Broeck [11], but in which there was a boundary beneath the sink sloping away without bound. These solutions again occurred at a unique Froude number for each angle. The result, when the angle of the wall is 30° downward, matches an exact solution first obtained by Sautreaux [10], and subsequently by Craya [2].

However, for two-dimensional withdrawal, when the fluid is of finite depth, there is a range of flow rates over which the cusp solutions exist for identical geometry, and even the smallest of these flow rates is much larger than the critical drawdown values observed experimentally, *e.g.* [3, 5]. In addition, there are two branches of solution with a cusp, only one of which exists for values of $F_B < 1$. This subcritical branch, obtained by Vanden-Broeck and Keller [12], provides a unique solution for a given geometry provided $F_B < 1$, but only occurs when the line sink is a long distance above the bottom, *i.e.* greater than ≈ 60% of the total fluid height. This branch has the solution of Tuck and Vanden-Broeck [11] as a limiting case as the fluid depth increases and is therefore a likely candidate for the critical flow. Vanden-Broeck [14] and Hocking and Vanden-Broeck [15] examined this branch carefully and found that solutions with waves can be obtained over a narrow range either side of the waveless solutions.

'Stagnation point' solutions (see Figure 1), in which the free surface rises up to a stagnation point above the sink, were first obtained by Peregrine [16] and Hocking and Forbes [17].

Forbes and Hocking [18] found stagnation-point solutions that include the effects of surface tension. They were able to demonstrate some non-uniqueness in the solution domain. Further stagnation-point solutions, but in fluid of finite depth, were obtained by Mekias and Vanden-Broeck [19, 20] and Vanden-Broeck [21]. These solutions were found to contain waves on the surface right up to $F_B = 1$ in most cases. They also obtained stagnation-point solutions without waves for $F_B > 1$ up to a limiting value. Hocking and Forbes [22] also found solutions in the range $F_B < 1$, but were unable to obtain waves, and found their solutions limited at around $F_B = 0{\cdot}24$.

The uncertainty surrounding the details of these withdrawal flows makes it necessary to examine them in greater detail in order to obtain a clearer understanding of their behaviour, with the ultimate goal of understanding the process of critical drawdown. In this paper we will concentrate on the subcritical ($F_B < 1$) solutions which contain a stagnation point on the free surface and which are influenced by the effects of surface tension. The regions in parameter space in which such solutions exist will be mapped and related to the other known solutions. The results provide an interesting insight into the nature of these withdrawal problems. No attempt will be made to look at the full unsteady problem, which has been considered recently by other researchers [23, 24].

In Section 2 of this paper we will formulate the problem as a nonlinear, singular integral equation. In Section 3 we briefly examine the behaviour near to the stagnation point and Section 4 describes the method of numerical solution. The results are described and discussed in Sections 5 and 6.

2. Problem formulation

The steady, irrotational motion of an inviscid, incompressible fluid due to a submerged sink is to be examined. The flow is assumed to be two-dimensional and gravity is acting vertically downwards (see Figure 1), Surface tension is assumed to be acting on the free surface, We concentrate in this paper on subcritical flows for which waves can be present on the free surface, and for which the free surface is horizontal directly above the sink. H is defined to be the height of the stagnation level of the free surface, and $H_B = H(1-\gamma)$ is the depth upstream in the fluid. If waves are present, this is taken to be the average depth. The average velocity U in the far field is then defined by $U = m/H$. We choose Cartesian coordinates so that the bottom is at $y = -H$, and the sink is at $x = 0$, $y = -H + H_S$. The stagnation point is at $x = 0$, $y = 0$ provided the surface tension is zero, but if the surface tension is not zero, the level of the free surface at the stagnation point may drop below $y = 0$. In the formulation given here we will use the symmetry of the flow about $x = 0$, and solve for only the left-hand half of the flow.

Let $z = x + \mathrm{i}y$ be the physical plane (see Figure 2(a)). The mathematical problem is to find a complex potential $w = \phi(x, y) + \mathrm{i}\psi(x, y)$,which is analytic in the flow domain and satisfies the conditions of no flow across the solid boundaries. Without loss of generality, we choose $\psi = 0$ on the free surface and $\phi = 0$ at the stagnation point above the sink. The surface of the water must also be at constant pressure, a condition provided by Bernoulli's equation, which, if we nondimensionalise with respect to the velocity $U = m/H$ and the depth H, takes the form

$$\eta + \tfrac{1}{2}F_{SP}^2 q^2 - \frac{\beta\eta''}{(1+\eta'^2)^{3/2}} = 0, \qquad (1)$$

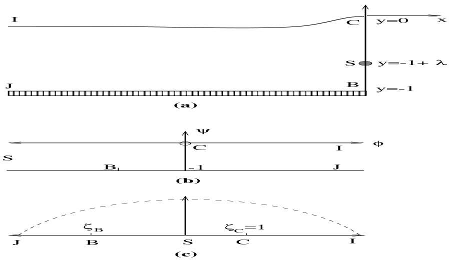

Figure 2. Mapped planes used in the problem formulation; (a) the physical z-plane, (b) the complex velocity potential w-plane, and (c) the upper-half ζ-plane.

where $y = \eta(x)$ is the equation for the elevation of the free surface, q is the fluid velocity, and F_{SP} is the Froude number defined above. The final term on the left side represents surface tension, where $\beta = T/(\rho g H^2)$ is a nondimensional surface-tension parameter (T being the dimensional value of the surface tension). Note that the upstream depth of the fluid is $(1-\gamma)$ and the upstream velocity is $1/(1-\gamma)$, so that the dimensionless flux is one. The relative height of the sink to the upstream depth is $\lambda = H_S/H_B = H_S/(H(1-\gamma))$.

Since the only term involving the velocity is squared, the equations are independent of the direction of the flow and therefore solutions are equally valid for a flow into (sink) or out (source) of the slot. However, we will discuss the flow as if it were into a sink.

To derive an integral equation for this problem we follow a procedure similar to that used in Hocking [9]. The transformation

$$e^{-\pi w} = \zeta \tag{2}$$

maps the infinite strip between $\psi = -1$ and $\psi = 0$ in the w-plane to the upper-half of the ζ-plane as shown in Figure 2(b). The point $\zeta = 0$ corresponds to the location of the sink, so that the free surface CI lies along the real ζ-axis where $\zeta \geqslant 1$. The line between $\zeta = 0$ and $\zeta = 1$ corresponds to the vertical wall above the sink, or to the line of symmetry of the flow if both sides are being considered.

The points between $\zeta = 0$ and $\zeta = \zeta_B$ correspond to the vertical wall beneath the sink, *i.e.* the line of symmetry of the flow, and $\zeta < \zeta_B$ to the bottom of the channel, which goes away horizontally to $x = -\infty$. The flow domain is the upper half ζ-plane (see Figure 2(c)). The case in which $\zeta_B = 0$ corresponds to the case of a line source or sink on the bottom of the channel.

We define a new function $\Omega(\zeta) = \delta(\zeta) + i\tau(\zeta)$, related to the complex conjugate of the velocity field by

$$w'(z(\zeta)) = u - iv = \left(\frac{1}{1-\gamma}\right) e^{-i\Omega(\zeta)}, \tag{3}$$

where the prime denotes differentiation with respect to z, and u and v are the horizontal and vertical components of fluid velocity.

The magnitude of the velocity at any point is then given by $|w'(z)| = e^{\tau(\zeta)}/(1-\gamma)$, and the angle any streamline makes with the horizontal is $\delta(\zeta)$. Since the total flux is one, and the upstream fluid depth approaches $(1-\gamma)$, $\tau \to 0$ as $|\zeta| \to \infty$, *i.e.* the upstream velocity approaches $1/(1-\gamma)$.

On the solid boundaries of the flow domain, the flow must follow the walls, so we choose $\delta(\zeta)$ to be the angle of the wall to the horizontal, that is

$$\delta(\zeta) = \begin{cases} 0 & \text{if } -\infty < \zeta < \zeta_B, \\ \pi/2 & \text{if } \zeta_B < \zeta < 0, \\ -\pi/2 & \text{if } 0 < \zeta \leqslant 1, \\ \text{unknown} & \text{if } 1 < \zeta < \infty. \end{cases} \tag{4}$$

Note that δ is unknown on $1 < \zeta < \infty$ because this is the free surface.

Following the procedure presented in [9], we can apply Cauchy's integral formula to $\Omega(\zeta)$ on a path consisting of the real ζ-axis, a semi-circle of radius $|\zeta| \to \infty$ in the upper-half plane, and a circle of vanishing radius about the point ζ. Substituting the known values of δ and taking the imaginary part, we obtain

$$\tau(\zeta) = \tfrac{1}{2}\ln\left[\frac{(\zeta-1)(\zeta-\zeta_B)}{\zeta^2}\right] - \frac{1}{\pi}\,⨍_1^\infty \frac{\delta(\zeta_0)}{\zeta_0-\zeta}\,\mathrm{d}\zeta_0. \tag{5}$$

The equation for constant pressure on the free surface, which we can obtain by combining Equations (1), (2) and (3) on $1 \leqslant \zeta < \infty$ is

$$\eta(1) + \frac{(1-\gamma)}{\pi}\int_1^\zeta \frac{\mathrm{e}^{-\tau(\zeta_0)}\sin\delta(\zeta_0)}{\zeta_0}\,\mathrm{d}\zeta_0 + \tfrac{1}{2}F_{SP}^2\frac{\mathrm{e}^{2\tau(\zeta)}}{(1-\gamma)^2} + \frac{\beta\pi\zeta\,\mathrm{e}^{\tau(\zeta)}}{1-\gamma}\frac{\mathrm{d}\delta}{\mathrm{d}\zeta} = 0. \tag{6}$$

Using (5) and (6) on the free surface, we get a singular nonlinear integral equation for $\delta(\zeta)$ on $1 \leqslant \zeta < \infty$. Using δ and τ, we can integrate (3) to obtain the location of points on the free surface. These may be written as

$$x(\zeta) = -\frac{(1-\gamma)}{\pi}\int_1^\zeta \frac{\mathrm{e}^{-\tau(\zeta_0)}\cos\delta(\zeta_0)}{\zeta_0}\mathrm{d}\zeta_0, \tag{7a}$$

and

$$\eta(\zeta) = \eta(1) - \frac{(1-\gamma)}{\pi}\int_1^\zeta \frac{\mathrm{e}^{-\tau(\zeta_0)}\sin\delta(\zeta_0)}{\zeta_0}\mathrm{d}\zeta_0. \tag{7b}$$

Note that, if there is no surface tension ($\beta = 0$), then $\eta(1) = 0$, but if $\beta \neq 0$, then Equation (1) gives that

$$\eta(1) = \frac{\beta\eta''(1)}{(1+\eta'(1)^2)^{3/2}}. \tag{8}$$

3. Behaviour near the stagnation point

An interesting aside is that for $F_{SP} = 0$, Equation (1) provides an exact solution, for any value of β, of the form $\eta(s) = -\beta^{1/2}\sin\theta_0\,\mathrm{e}^{-\beta^{-1/2}s}$ where θ_0 is the angle of the free surface at the stagnation point, and s is the arclength along the free surface. In this situation, with zero flow velocity, these solutions represent a meniscus at the attachment point, and they can have any angle, θ_0. This leads us to question whether this is possible if the flow rate is not zero. Tuck and Vanden Broeck [11] showed that if $\beta = 0$ the free surface must be perpendicular to the wall at the attachment point.

In the variables defined above, we can differentiate Equation (6) with respect to ζ, giving

$$\frac{(1-\gamma)\,\mathrm{e}^{-\tau(\zeta)}\sin\delta(\zeta)}{\pi\zeta} + \frac{F_{SP}^2\,\mathrm{e}^{2\tau(\zeta)}\tau'(\zeta)}{(1-\gamma)^2} + \frac{\beta\pi\,\mathrm{e}^{\tau(\zeta)}}{(1-\gamma)}[\delta'(\zeta) + \zeta\delta''(\zeta) + \zeta\tau'(\zeta)\delta'(\zeta)] = 0. \quad (9)$$

Multiplying by $\mathrm{e}^{\tau}\zeta\pi/(1-\gamma)$ gives

$$\sin\delta(\zeta) + \pi F_{SP}^2\zeta\frac{\mathrm{e}^{3\tau(\zeta)}}{(1-\gamma)^3}\tau'(\zeta) + \beta\zeta\left(\frac{\pi}{1-\gamma}\right)^2\mathrm{e}^{2\tau(\zeta)}[\delta'(\zeta) + \zeta\delta''(\zeta) + \zeta\tau'(\zeta)\delta'(\zeta)] = 0. \quad (10)$$

Noting that if $\zeta \to 1$, and if the angle at the stagnation point is δ_S, we have

$$\tau(\zeta) \to (\zeta - 1)^{1/2+\delta_S/\pi}$$

and

$$\tau'(\zeta) \to \frac{\frac{1}{2} + \delta_S/\pi}{(\zeta - 1)},$$

so that in the limit as $\zeta \to 1^+$,

$$\sin\delta_S + \frac{\pi F_{SP}^2}{(1-\gamma)^3}(\tfrac{1}{2} + \delta_S/\pi)(\zeta - 1)^{(1/2+3\delta_S/\pi)} + \frac{\beta\pi^2}{(1-\gamma)^2}[(\delta'(1) + \delta''(1))(\zeta - 1)^{1+2\delta_S/\pi} + \delta'(1)(\tfrac{1}{2} + \delta_S/\pi)(\zeta - 1)^{2\delta_S/\pi}] \approx 0. \quad (11)$$

This equation only has finite solutions if $\delta_S \geqslant 0$, and, if $\delta_S > 0$, then all terms are zero, so that we must have $\sin\delta_S = 0$. Thus, it seems that, even with surface tension in the equations, the free surface must be perpendicular at the attachment point. Note that the case $F_{SP} = 0$ is a singular limit in this formulation, because we can no longer use the mapping (3) in the same way. That is why we do not recover the exact solution given above when $F_{SP} = 0$.

4. Numerical solution

No closed-form solution is known for the full nonlinear system of equations given by (5) and (6) (except for the case $F = \infty$, $\beta = 0$, see *e.g.* [12], and $F = 0$ as described above), but it can be solved quite well by means of collocation.

The numerical scheme is very similar to that of [15], and so we only briefly describe it here. The behaviour of ζ is like e^{ϕ}, so we choose to make the transformation $\zeta = \mathrm{e}^{\alpha}$.

Points were chosen at N equally spaced values of α up to α_t, a truncation at sufficiently large $\zeta_t = e^{\alpha_t}$. The integral equation consisting of (5) and (6) was evaluated at the mesh points α_j, for $j = 2, 3, \ldots, N$, giving $N - 1$ equations for the unknown values δ_j, $j = 2, 3, \ldots, N$ and the free-stream depth upstream γ. The value of δ_1 is known to be zero, since the free surface is assumed to be horizontal at the stagnation point. Thus, we have $N - 1$ equations for the N unknowns. We can ensure that the distance from the bottom to the stagnation point is correct by using a variation on Equation 7(b),

$$1 - \eta(1) = -\frac{(1-\gamma)}{\pi} \int_{\zeta_B}^{1} \frac{e^{-\tau(\zeta_0)} \sin \delta(\zeta_0)}{\zeta_0} \, d\zeta_0. \tag{12}$$

This is now a closed system of N equations for N unknowns and, given an initial guess, we can solve this using a Newton–Raphson iteration scheme. However, in this formulation the sink depth comes out of the solution as an output. This turns out to be important later, but on some occasions it is desirable for us to be able to specify the sink depth, and we may do this by including an extra equation similar to (12), but for the sink hight only, and making ζ_B the extra unknown.

Care must be taken in evaluating the Cauchy Principal Value integral in Equation (5), as described in [15]. A trapezoidal-rule integration scheme was found to be sufficient for all calculations, and a step size of around $\Delta\alpha = 0{\cdot}05$ was required for us to get solutions accurate to two-to-three figure accuracy using around 400 points on the surface.

Given a reasonable starting guess for δ and γ, this scheme converged rapidly, usually taking only 4 or 5 iterations. The initial guess $\delta(\zeta) = 0$, $\gamma = 0$ was good enough for most situations, especially small values of F_{SP}. Once a solution was obtained for a particular case, it was used as a starting guess for other cases, for example for increasing values of the Fronde number. We tested the method by comparing the solutions with those obtained by Mekias and Vanden-Broeck [20] and Vanden-Broeck [21] with zero surface tension.

5. Results

Solutions were computed for a range of values of Froude number, surface tension and sink height. Generally, the solutions were found to depend on all three parameters. In cases where surface tension was small, the solutions included waves on the free surface as F_B was increased, very similar to those of Mekias and Vanden-Broeck [20]. However, as the surface tension was increased, the waves decreased in amplitude and eventually disappeared completely, leaving a waveless free surface for larger values of surface tension. Figure 3 shows two typical solutions with the same values of F_B and λ (sink height) for different values of surface tension. The effect mentioned is clear in this case.

In almost all cases solutions were obtained up to a limiting depth-based Froude number F_B. Table 1 shows the limiting value of F_B for different sink heights for $\beta = 0{\cdot}1$, typical of the behaviour at all values of β. Figure 4 shows the limiting values of Froude number for different sink heights at different values of the surface tension. In most cases this limit was close to one, but it is clear that, when λ, the sink height, is close to one, the limiting value for all values of surface tension (including zero) drops dramatically to sit very close

Table 1. Maximum computed Froude number for varying sink depths with $\beta = 0.1$.

ζ_R	F_B(max)	$\lambda = h_S/h_B$
$-0{\cdot}005$	0·849	0·056
$-0{\cdot}050$	0·849	0·174
$-0{\cdot}500$	0·849	0·473
$-4{\cdot}000$	0·617	0·609
$-30{\cdot}00$	0·246	0·727
$-600{\cdot}0$	0·075	0·839

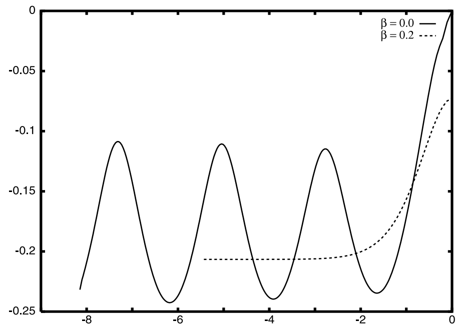

Figure 3. Free-surface profiles for two cases with very similar Froude numbers, $F_B = 0{\cdot}728$, $0{\cdot}722$, and sink height, $\lambda = 0{\cdot}2$, but different values of surface tension $\beta = 0$, $0{\cdot}2$. Surface tension has caused the level of the stagnation point to drop, and completely damped out the waves.

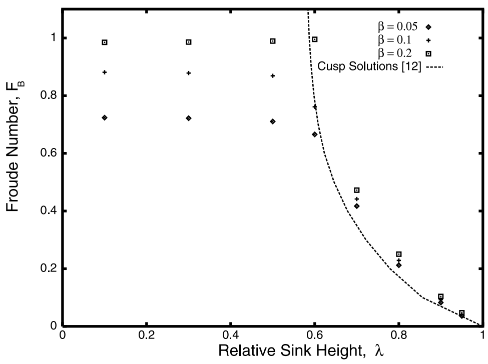

Figure 4. Limiting Froude numbers, F_B for different values of surface tension $\beta = 0{\cdot}05$, $0{\cdot}1$, $0{\cdot}2$ and sink height λ. The dashed line is the waveless solutions of Vanden-Broeck and Keller [29].

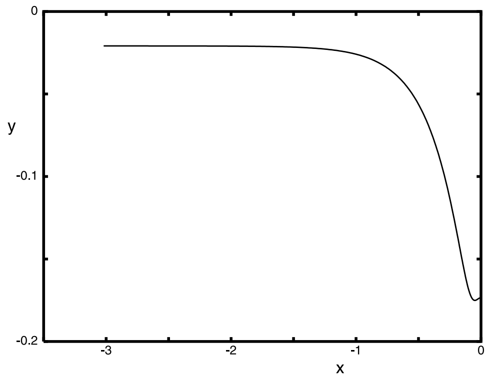

Figure 5. Free-surface shape near the limiting form for sink height $\lambda = 0{\cdot}778$, surface tension $\beta = 0{\cdot}05$ and Froude number $F_B = 0{\cdot}2064$.

to the curve depicting the cusp solutions of Vanden-Broeck and Keller [12]. Those solutions which do contain waves appear to steepen very quickly close to the limiting value. As one expects with nonlinear waves, the troughs get broader and the crests get narrower. The peaks appear to sharpen, but one can only speculate that they form a 120° corner as they reach a breaking-wave height. Those which approach the horizontal limits (the curve of Vanden-Broeck and Keller [12]) depicted in Figure 4 do not appear to behave in this way. There is thus a fundamental difference between the two kinds of solution. This highlights the importance of the cusp solutions of Vanden-Broeck and Keller [12] when considering the critical drawdown condition. The region in Figure 4 bounded approximately by $F_B = 1$ and this curve would seem to be the only region in which stagnation point solutions of any kind can exist.

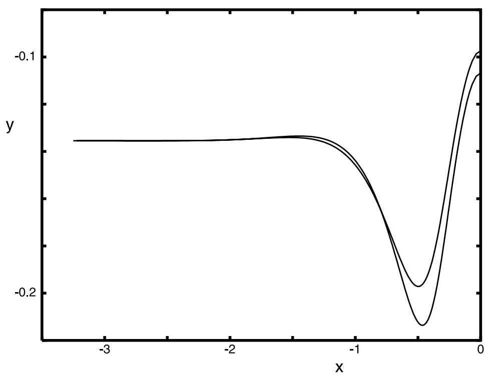

Figure 6. Two solutions for identical Froude number, $F_B = 0{\cdot}560$, sink height, $\lambda = 0{\cdot}654$ and surface tension, $\beta = 0{\cdot}05$, demonstrating the non-uniqueness.

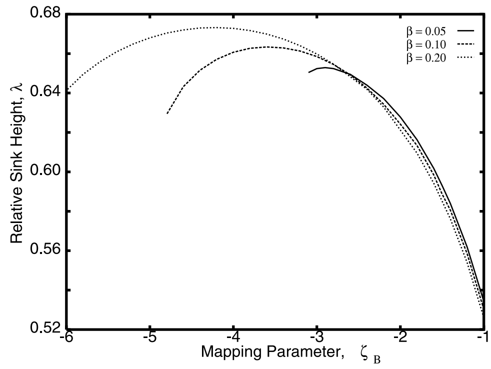

Figure 7. Plot of sink height λ against the mapping parameter ζ_B for three different values of $\beta = 0{\cdot}05$, $0{\cdot}1$, $0{\cdot}2$ showing how the non-uniqueness shown in Figure 6 arises. The Froude number is $F_B = 0{\cdot}560$ for all cases.

Figure 5 shows a solution obtained for a sink height of $\lambda = 0{\cdot}778$, surface tension of $\beta = 0{\cdot}05$ and Froude number of $F_B = 0{\cdot}2064$, in which the shape of the free surface appears to be like a cusp solution, but is prevented from being so by surface tension (causing horizontal attachment above the sink). The implications of this will be discussed later, but there is one other aspect of the solutions which is worthy of consideration.

The mapping parameter ζ_B, which corresponds to the point on the bottom beneath the sink, plays a significant role in determining the height of the sink. Increasing the magnitude of ζ_B generally moves the sink further off the bottom. However, it was found that increasing the magnitude sometimes resulted in the sink height increasing to some maximum, and then turning back and decreasing again while Froude number and surface tension remained constant. This means that there is a non-uniqueness in the solutions, *i.e.* that for the same Froude number, surface tension and sink height there is more than one solution. Figure 6 shows a plot of the two free surface shapes obtained for one such case, in which $F_B = 0{\cdot}56$, $\lambda = 0{\cdot}654$ and $\beta = 0{\cdot}05$.

Figure 7 shows a plot of how this non-uniqueness arises, with the parameter ζ_B plotted against sink height λ for three different values of surface tension, $\beta = 0{\cdot}05$, $0{\cdot}1$, $0{\cdot}2$. In all cases the Froude number is $F_B = 0{\cdot}56$. This non-uniqueness was noted before in the work of Forbes and Hocking [18], and so should not be regarded as surprising, but does confirm its existence when surface tension is present.

6. Concluding remarks

In this paper, we have used a boundary-integral-equation method to compute numerical solutions to the problem of steady withdrawal from a layer of finite depth through a line sink in which surface tension is acting on the free surface. It is shown that solutions appear to exist over almost all of the parameter space with Froude number less than unity, except for values close to one for some values of surface tension, and for a region which appears to be bounded by the cusp solutions of Vanden-Broeck and Keller [12]. Figure 4 clearly shows these regions. When this subcritical region is considered, it appears that all regions in which it is possible to get solutions have solutions of the stagnation-point type, except along a single limiting curve. If the cusp solutions on this limiting curve do correspond to the critical drawdown values, then it would not be possible to find single-layer flow solutions in the region above this curve and below $F_B = 1$.

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